

ANALYTIC INTEGRABILITY OF A CLASS OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

INTÉGRABILITÉ ANALYTIQUE D’UNE CLASSE DE SYSTÈMES DIFFÉRENTIELS POLYNÔMIAUX DANS LE PLAN

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ABSTRACT. In this paper we find necessary and sufficient conditions in order that the differential systems of the form $\dot{x} = xf(y)$, $\dot{y} = g(y)$, with f and g polynomials, have a first integral which is analytic in the variable x and meromorphic in the variable y . We also characterize their analytic first integrals in both variables x and y .

These polynomial differential systems are important because after a convenient change of variables they contain all quasi-homogeneous polynomial differential systems in \mathbb{R}^2 .

RÉSUMÉ. Dans cet article, nous trouvons des conditions nécessaires et suffisantes pour que les systèmes différentiels de la forme $\dot{x} = xf(y)$, $\dot{y} = g(y)$, avec f et g polynômes, ont une première intégrale qui est analytique dans la variable x et méromorphe dans la variable y . Nous caractérisons aussi leur intégrales première analytique dans les deux variables x et y .

Ces systèmes différentiels polynômiaux sont importants parce que, après un changement convenable de variables ils contiennent tous les systèmes différentiels polynômiaux quasi-homogènes en \mathbb{R}^2 .

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let \mathbb{C} be the set of complex numbers and $\mathbb{C}[y]$ the ring of all polynomials in the variable y with coefficients in \mathbb{C} . In this paper we consider the polynomial differential systems of the form

$$(1) \quad \dot{x} = xf(y), \quad \dot{y} = g(y),$$

where $f, g \in \mathbb{C}[y]$ and are coprime. The dot denotes the derivative with respect to the independent variable t real or complex. We denote by $\mathcal{X} = (xf(y), g(y))$ the polynomial vector field associated to system (1), and we say that the degree of the system is $n = \max\{\deg xf(y), \deg g(y)\}$. For the sake of simplicity, we assume for the rest of the paper that system (1) is not linear, that is $n > 1$.

We recall that given a planar polynomial differential system (1), we say that a function $H: \mathcal{U} \subset \mathbb{C}^2 \rightarrow \mathbb{C}$ with \mathcal{U} an open set, is a *first integral* of system (1) if

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H is continuous, not locally constant and constant on each trajectory of the system contained in \mathcal{U} . We note that if H is of class at least C^1 in \mathcal{U} , then H is a first integral if it is not locally constant and

$$xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y} = 0$$

in \mathcal{U} . We call the *integrability problem* the problem of finding such a first integral and the functional class where it belongs. We say the system has an *analytic first integral* if there exists a first integral $H(x, y)$ which is an analytic function in the variables x and y . We say that the system has a *pseudo-meromorphic first integral* if there exists a first integral $H(x, y)$ which is an analytic function in the variable x and a meromorphic function in the variable y .

The aim of this paper is to characterize the existence of first integrals of system (1) that can be described by functions that are analytic or pseudo-meromorphic.

Let α_l for $l = 1, \dots, k$ be the zeros of g . We say that g is *square-free* if $g(y) = \prod_{l=1}^k (y - \alpha_l)$ with $\alpha_l \neq \alpha_j$ for $l, j = 1, \dots, k$ and $l \neq j$. When g is square-free we define $\gamma_l = f(\alpha_l)/g'(\alpha_l)$ for $l = 1, \dots, k$. With this notation we introduce the main result of the paper.

Theorem 1. *System (1) has a pseudo-meromorphic first integral if and only if $g(y)$ is square-free. Moreover, if $\gamma_l > 0$ for all $l = 1, \dots, k$ then the first integral is analytic, otherwise it is a pseudo-meromorphic function with poles on $y = \alpha_l$ if $\gamma_l > 0$.*

The proof of Theorem 1 is given in section 2. Furthermore, the specific form of the first integral is given in the proof of Theorem 1.

Example 2. *Consider the differential system*

$$\dot{x} = xy^3, \quad \dot{y} = y + 1.$$

This system has the analytic first integral

$$H(x, y) = e^{-(y+1)(2y^2-5y+11)/6} x(y+1).$$

Note that $g(y) = y + 1$ is square-free, $\alpha_1 = -1$ and $\gamma_1 = -1 < 0$.

Example 3. *Consider the differential system*

$$\dot{x} = xy^3, \quad \dot{y} = y - 1.$$

This system has the pseudo-meromorphic first integral

$$H(x, y) = \frac{e^{(1-y)(2y^2+5y+11)/6} x}{y-1}.$$

Note that $g(y) = y - 1$ is square-free, $\alpha_1 = 1$ and $\gamma_1 = 1 > 0$.

System (1) is of separate variables and appears in many situations. In Lemma 2.2 of [1] it is proved that there exists a blow-up change of variables that transforms any quasi-homogeneous polynomial differential system into a differential system (1). However we point out that not all the planar polynomial differential systems (1) come

from quasi-homogenous polynomial differential systems. We recall that a polynomial differential system

$$\dot{x} = P(x, y) \quad \dot{y} = Q(x, y)$$

is quasi-homogeneous if there exists $s_1, s_2, d \in \mathbb{N}$ (here \mathbb{N} denotes the set of positive integers) such that for arbitrary $\alpha \in \mathbb{C}$,

$$(2) \quad P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x, y), \quad Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x, y).$$

From Theorem 3.1b) of [1] and Proposition 1 of [3] it follows the next result.

Theorem 4. *The quasi-homogeneous polynomial differential system (2) has an analytic first integral if and only if $g(y)$ is square-free, $\deg f < \deg g$ and $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \dots, k$ and $1 + \gamma_1 + \gamma_2 + \dots + \gamma_k \geq 0$.*

Note that Theorem 1 extends the result of Theorem 4 only valid for the quasi-homogeneous polynomial differential systems. Recall that quasi-homogeneous polynomial differential systems can be written as a subclass of the polynomial differential systems (1) using the Lemma 2.2 of [1].

2. PROOF OF THEOREM 1

Assume that system (1) has a pseudo-meromorphic first integral. Then it can be written as a power series in x in the form

$$H(x, y) = \sum_{k \geq 0} a_k(y)x^k,$$

where $a_k(y)$ is a meromorphic function in the variable y . Then, it must satisfy

$$(3) \quad xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y} = 0,$$

that is

$$0 = \sum_{k \geq 0} kf(y)a_k(y)x^k + \sum_{k \geq 0} g(y)a'_k(y)x^k = \sum_{k \geq 0} (kf(y)a_k(y) + g(y)a'_k(y))x^k.$$

Hence,

$$a'_0(y) = 0 \quad \text{that is} \quad a_0(y) = \text{constant}$$

and for $k \geq 1$,

$$(4) \quad kf(y)a_k(y) + g(y)a'_k(y) = 0 \quad \text{that is} \quad \frac{a'_k(y)}{a_k(y)} = \frac{-kf(y)}{g(y)}.$$

If $\deg f \geq \deg g$ and we consider the division of $-kf(y)$ by $g(y)$ we can write

$$kf(y) = q(y)g(y) + r(y),$$

where $r(y)$ cannot be zero taking into account that f and g are coprime and $\deg \psi < \deg g$. Hence equation (4) takes the form

$$(5) \quad \frac{a'_k(y)}{a_k(y)} = -q(y) - \frac{r(y)}{g(y)}.$$

Integrating this equation we have

$$(6) \quad a_k(y) = C e^{-Q(y)} e^{-\int \frac{r(v)}{g(v)} dv}$$

where C is a constant of integration and $Q'(y) = q(y)$. Therefore, since the first factor of (6) is an analytic function, we must study the second factor in (6).

Assume that g is not square free. Using an affine transformation of the form $z = y + \alpha$ with $\alpha \in \mathbb{C}$ if it is necessary, we can assume that z is a multiple of g , that is, $\tilde{g}(z) = z^m R(z)$, where $\tilde{g}(z) = g(z - \alpha)$ with $m > 1$ an integer and $R(0) \neq 0$. Since f and g are coprime we also have $\tilde{r}(0) \neq 0$, where $\tilde{r}(z) = r(z - \alpha)$. Now we develop $\tilde{r}(z)/\tilde{g}(z)$ in simple fractions of z , that is,

$$\frac{\tilde{r}(z)}{\tilde{g}(z)} = \frac{c_m}{z^m} + \frac{c_{m-1}}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)},$$

where $\alpha(z)$ is a polynomial with $\deg \alpha(z) < \deg R(z)$, and $c_i \in \mathbb{C}$ for $i = 1, \dots, m$. Note that $c_m \neq 0$. Therefore integrating this last expression we have

$$\exp \left(\int \frac{\tilde{r}(z)}{\tilde{g}(z)} dz \right) = \exp \left(\frac{c_m}{(1-m)z^{m-1}} \right) \cdot \exp \left(\int \left(\frac{c_{m-1}}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)} \right) dz \right).$$

Note that the first exponential factor cannot be simplified by any part of the second exponential factor. Moreover $c_m \neq 0$ and we get a contradiction with the fact that the left hand side must be a meromorphic function in the variable y while $\exp(c_m/((1-m)z^{m-1}))$ has an essential singularity at $z = 0$, and this it is not meromorphic in z . Therefore, we conclude that $g(y)$ is square-free. Hence we write

$$\frac{r(z)}{g(z)} = \frac{\gamma_1}{z - \alpha_1} + \cdots + \frac{\gamma_k}{z - \alpha_k}.$$

Then,

$$\int \frac{r(z)}{g(z)} dz = \sum_{j=1}^k \int \frac{\gamma_j}{z - \alpha_j} dz = \sum_{j=1}^k \gamma_j \log(z - \alpha_j)$$

and, consequently,

$$e^{\int \frac{r(z)}{g(z)} dz} = \prod_{j=1}^k (z - \alpha_j)^{\gamma_j}.$$

Note that this expression is always a meromorphic function. If $\gamma_j > 0$ for all $j = 1, \dots, k$ then it is an analytic function in the variable y , otherwise it is meromorphic with poles on the α_j such that $\gamma_j < 0$. Hence $a_k(y)$ is an analytic function in y if $\gamma_j < 0$ for $j = 1, \dots, k$, and it is meromorphic with poles on the α_j with $\gamma_j > 0$.

Conversely, assume that g is square-free and that $f(y) = q(y)g(y) + r(y)$. We will show that

$$H(x, y) = x e^{-\int q(y) dy} (y - \alpha_1)^{-\gamma_1} \cdots (y - \alpha_k)^{-\gamma_k},$$

with $\gamma_i = r(\alpha_i)/g'(\alpha_i)$ for $i = 1, \dots, k$ is a pseudo-meromorphic function, and it is analytic if all $\gamma_j < 0$ for $j = 1, \dots, k$. Now we must show that indeed it is a first

integral of system (1). We set $\phi(y) = (y - \alpha_1)^{\gamma_1} \cdots (y - \alpha_k)^{\gamma_k}$. Note that

$$\begin{aligned} 0 &= xf(y) \frac{\partial H}{\partial x} + g(y) \frac{\partial H}{\partial y} \\ &= xf(y) e^{-\int q(y) dy} \phi(y) + xg(y) (-q(y)\phi(y) - \phi'(y)) e^{-\int q(y) dy} \\ &= xe^{-\int q(y) dy} (f(y)\phi(y) - g(y)q(y)\phi(y) - g(y)\phi'(y)) \\ &= xe^{-\int q(y) dy} (r(y)\phi(y) - g(y)\phi'(y)). \end{aligned}$$

To see that this last expression is identically zero it is equivalent to see that

$$\frac{\phi'(y)}{\phi(y)} = \frac{r(y)}{g(y)}.$$

Recalling the expression of $\phi(y)$ we have

$$\frac{\phi'(y)}{\phi(y)} = \frac{\gamma_1}{y - \alpha_1} + \frac{\gamma_2}{y - \alpha_2} + \cdots + \frac{\gamma_k}{y - \alpha_k}.$$

Taking common denominator and recalling that $g(y) = c(y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_k)$ we obtain

$$\frac{\phi'(y)}{\phi(y)} = \frac{c}{g(y)} \sum_{i=1}^k \gamma_i \prod_{j=1, j \neq i}^k (y - \alpha_j).$$

Now substituting the values of $\gamma_i = r(\alpha_i)/g'(\alpha_i)$ and taking into account that

$$g'(\alpha_i) = c \prod_{j=1, j \neq i}^k (\alpha_i - \alpha_j),$$

we get

$$(7) \quad \frac{\phi'(y)}{\phi(y)} = \frac{1}{g(y)} \sum_{i=1}^k r(\alpha_i) \prod_{j=1, j \neq i}^k \frac{y - \alpha_j}{\alpha_i - \alpha_j} = \frac{r(y)}{g(y)}.$$

The last expression in the sum, recalling that $\deg r < \deg g$, is the expression of the Lagrange polynomial which interpolates $r(y)$ in the k points $(\alpha_i, r(\alpha_i))$, for $i = 1, 2, \dots, k$, see for more details [2]. Therefore this polynomial is $r(y)$, and we conclude that the expression (7) is satisfied. This completes the proof of the theorem.

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